

The Calogero equation and Liouville type equations

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Abstract. In this paper we present a two-component generalization of the C-integrable Calogero equation (see [1]). This system is C-integrable as well, and moreover we show that the Calogero equation and its two-component generalization are solvable by a reciprocal transformation to ODE's. Simultaneously we obtain a generalized Liouville equation (34), determined by two arbitrary functions of one variable.

1 Introduction

F.Calogero in his article "A solvable nonlinear wave equation" presented the nonevolutionary equation

$$u_{xt} = uu_{xx} + \Phi(u_x), \quad (1)$$

with an arbitrary function $\Phi(p)$. Moreover he have found a general solution, but in some complicated form. Here we suggest a new approach for description of this solution in more clear form by using suitable reciprocal transformation. In result we obtain a general solution for Liouville type equations (see below).

At first we notice that the Calogero equation (1) has a special conservation law

$$\partial_t F(u_x) = \partial_x [uF(u_x)], \quad (2)$$

where

$$F(p) = \exp\left[\int \frac{pdp}{\Phi(p)}\right]. \quad (3)$$

The corresponding reciprocal transformation

$$dz = F(p)dx + uF(p)dt, \quad dy = dt \quad (4)$$

yields recalculation of independent variables

$$\partial_x = v\partial_z \quad \text{and} \quad \partial_t = \partial_y + uv\partial_z, \quad (5)$$

where $v = F(p)$. Then the conservation law (2) has form

$$u_z = \partial_y(-1/v) \quad (6)$$

and the Calogero equation (1) transforms into

$$p_y = \Phi(p), \quad (7)$$

where (see (5) and (6))

$$p = u_x = vu_z = \partial_y \ln v. \quad (8)$$

Thus, a solution of the Calogero equation (1) can be presented in implicit form (see (7)) and found consequently by integration from (7), (8) and from the inverse reciprocal transformation

$$dx = \frac{1}{F(p)}dz - udy, \quad dt = dy. \quad (9)$$

In second section we present couple well-known examples of such integrable equations. In third section we describe another reciprocal transformation from the generalized Hunter-Saxton equation into the Liouville equation. In fourth section we discuss general reciprocal transformations from a nonlinear PDE more complicated than the Calogero equation (1) into an ODE of the second order. In last fifth section we construct a two-component generalization of the Calogero equation (1) and present its solution by corresponding reciprocal transformation to ODE of the second order. Thus, the major aim of this article is possibility to demonstrate that some C-integrable equations (or systems) can be integrated by suitable reciprocal transformations into corresponding ODE's.

2 Remarkable examples

One of them is the generalized Hunter-Saxton equation

$$u_{xt} = uu_{xx} + \varepsilon u_x^2 \quad (10)$$

was studied in [2], where a general solution was found by characteristic method for $\varepsilon = 1/2$. Exactly this case was also studied by P.Olver and P.Rosenau in them article [3] from another point of view. Here we construct a general solution for arbitrary ε . Since $\Phi(p) = \varepsilon p^2$, then

$$\begin{aligned} F(p) &= p^{1/\varepsilon}, \quad p = \frac{1}{A(z) - \varepsilon y}, \\ u &= B'(y) + \int [A(z) - \varepsilon y]^{(1-\varepsilon)/\varepsilon} dz, \\ x &= -B(y) + \int [A(z) - \varepsilon y]^{1/\varepsilon} dz. \end{aligned} \quad (11)$$

If $\varepsilon = 1/2$, then we obtain a general solution of the Hunter-Saxton equation

$$\begin{aligned} u &= B'(y) + \int A(z)dz - \frac{1}{2}yz, \\ x &= -B(y) + \int A^2(z)dz - y \int A(z)dz + \frac{1}{4}y^2z. \end{aligned} \tag{12}$$

If $\varepsilon = 1$, equation (10) allows the reduction

$$u_t = uu_x$$

Its solution is

$$x = -B(y) + C(z) - yz, \quad u = B'(y) + z$$

It is easy to check that in this case (just one equation of the first order) $B(y) = \alpha y + \beta$ and finally we obtain the standard solution

$$(x + \beta) + ut = C(u - \alpha)$$

At this conference NEED's 2000 in Gukova (Turkey) one of participants professor Valery Druma demonstrated for author another nonevolutionary equation appeared in intersection of projective geometry and gravity (see [4])

$$u_{xxt} = uu_{xxx}.$$

This equation has obvious integral

$$u_{xt} = uu_{xx} - \frac{1}{2}u_x^2 + \gamma(t).$$

For special case $\gamma = 0$ this is particular case of the generalized Hunter-Saxton equation (10).

3 The Calogero equation and The Liouville equation

The generalized Hunter-Saxton system (10) has another conservation law in comparison with (2)

$$\partial_t[(u_{xx})^{1/(2\varepsilon+1)}] = \partial_x[u(u_{xx})^{1/(2\varepsilon+1)}]. \tag{13}$$

The corresponding reciprocal transformation

$$dw = qdx + uqdt, \quad d\tau = dt \tag{14}$$

yields a recalculation of independent variables

$$\partial_x = q\partial_w, \quad \partial_t = \partial_\tau + uq\partial_w, \tag{15}$$

where

$$q = (u_{xx})^{1/(2\varepsilon+1)}. \quad (16)$$

Then the conservation law (13) transforms into

$$u_w = -\partial_\tau(1/q) \quad (17)$$

and simultaneously the generalized Hunter-Saxton equation transforms into

$$s_\tau = \varepsilon s^2, \quad (18)$$

where

$$s = u_x = qu_w = \partial_\tau \ln q. \quad (19)$$

Thus a general solution of the generalized Hunter-Saxton equation can be presented in implicit form (see (18)) and found consequently by integration from (18), (19) and from the inverse reciprocal transformation

$$dx = \frac{1}{q}dw - u d\tau, \quad dt = d\tau. \quad (20)$$

This solution is

$$x = \int \frac{dw}{B(w)} [A(w) + \varepsilon\tau]^{1/\varepsilon} + C(\tau), \quad u = - \int \frac{dw}{B(w)} [A(w) + \varepsilon\tau]^{(1-\varepsilon)/\varepsilon} - C'(\tau), \quad (21)$$

where

$$q = B(w)[A(w) + \varepsilon\tau]^{-1/\varepsilon}, \quad s = -[A(w) + \varepsilon\tau]^{-1}. \quad (22)$$

Simultaneously, anyone can recalculate (16)

$$q^{2\varepsilon+1} = u_{xx} = q\partial_w(qu_w), \quad (16a)$$

and substitute (19) into (16a) at next step. Then the Hunter-Saxton equation transforms by reciprocal transformation (14) into the Liouville equation

$$\partial_{w\tau} \ln q = q^{2\varepsilon}. \quad (23)$$

Substituting (22) into (23) anyone can find well-known general solution of the Liouville equation

$$q = [A'(w)]^{1/2\varepsilon} [A(w) + \varepsilon\tau]^{-1/\varepsilon}, \quad (24)$$

where

$$B(w) = [A'(w)]^{1/2\varepsilon}.$$

Thus finally the general solution of the Hunter-Saxton equation is

$$\begin{aligned} x &= \int [A'(w)]^{-1/2\varepsilon} [A(w) + \varepsilon\tau]^{1/\varepsilon} dw + C(\tau), \\ u &= - \int [A'(w)]^{-1/2\varepsilon} [A(w) + \varepsilon\tau]^{(1-\varepsilon)/\varepsilon} dw - C'(\tau). \end{aligned} \quad (21a)$$

If a general solution of the generalized Hunter-Saxton equation depends on two functions of one variable (see (21a) or (11)), a general solution of the Liouville equation must depend on two functions of one variable too. It is easy to reconstruct by using the obvious symmetry of the Liouville equation $\tau \rightarrow R(\tau)$, $q \rightarrow q[R'(\tau)]^{1/2\varepsilon}$ (see (24))

$$q = [A'(w)R'(\tau)]^{1/2\varepsilon} [A(w) + \varepsilon R(\tau)]^{-1/\varepsilon}. \quad (24a)$$

The particular case ($\varepsilon = 1/2$) was studied in [5] for high-frequency limit of Camassa-Holm equation. However, here we showed that the generalized Hunter-Saxton equation transformable into the Liouville equation. Thus, in this section we described relationship between the Hunter-Saxton equation and the Liouville equation, simultaneously we integrated as the Hunter-Saxton equation as the Liouville equation. However, in next section we present general scheme for integrability of the Calogero equation and we will show, that by this way the Calogero equation can be transform into more general equation than the Liouville equation, and simultaneously the Calogero equation can be integrated again by reciprocal transformation into ODE.

4 A General Case

We suggest that the nonevolutionary equation

$$u_{xt} = \varphi(u, u_x, u_{xx}) \quad (25)$$

has the special conservation law

$$\partial_t f(u_x, u_{xx}) = \partial_x [u f(u_x, u_{xx})]. \quad (26)$$

Then right-hand side of (25) is more determined

$$u_{xt} = uu_{xx} + \psi(u, u_x), \quad (25a)$$

where from compatibility condition of (25) and (26) we have

$$\psi \frac{\partial f}{\partial u_x} + \left[\frac{\partial \psi}{\partial u} u_x + \left(u_x + \frac{\partial \psi}{\partial u_x} \right) u_{xx} \right] \frac{\partial f}{\partial u_{xx}} = f u_x. \quad (27)$$

If we introduce the reciprocal transformation

$$dz = w dx + u w dt, \quad dy = dt \quad (28)$$

for (25a), where $w = f(u_x, u_{xx})$, then we obtain from (25a)

$$(w u_z)_y = \psi(u, w u_z) \quad (29)$$

and

$$u_z = w^{-2}w_y. \quad (30)$$

Thus, we finally obtain *ordinary differential equation*

$$\partial_y^2 \ln w = \psi(u, \partial_y \ln w) \quad (29a)$$

and

$$w = f(\partial_y \ln w, w \partial_{yz} \ln w). \quad (31)$$

However, the ODE (29a) can be integrated just if $\psi = \psi(u_x)$. Then the equation (27)

$$\psi \frac{\partial f}{\partial u_x} + [u_x + \psi'(u_x)] u_{xx} \frac{\partial f}{\partial u_{xx}} = f u_x. \quad (27a)$$

can be integrated immediately

$$f = \beta(u_x) \zeta\left(\frac{\beta(u_x) \psi(u_x)}{u_{xx}}\right), \quad (32)$$

where

$$\beta(\tau) = \exp\left[\int \frac{\tau d\tau}{\psi(\tau)}\right] \quad (33)$$

and $\zeta(v)$ is an arbitrary function. Thus the Calogero equation (1) by generalized conservation law (26) (see reciprocal transformation (28)) transforms into an **ordinary differential equation** (see (29) and (29a))

$$s_{yy} = \psi(s_y) \quad (29b)$$

and simultaneously into a hyperbolic equation (see (31))

$$s_{yz} = \psi(s_y) \alpha(\beta(s_y) e^{-s}), \quad (34)$$

where

$$s = \ln w, \quad \zeta\left(\frac{1}{\tau \alpha(\tau)}\right) = \tau. \quad (35)$$

Thus, we obtain following beautiful fact: every hyperbolic equation (34) with two arbitrary functions $\psi(\tau)$ and $\alpha(\tau)$ is C-integrable (see (29b)), if $\beta(\tau)$ is satisfying to (33). From this point of view this is natural generalization of the Liouville equation (23).

5 Two-component Generalization

The Calogero equation (1) allows the two-component generalization

$$\eta_t = \partial_x(u\eta), \quad u_{xt} = uu_{xx} + \psi(\eta, u_x). \quad (36)$$

By application of the reciprocal transformation

$$dz = \eta dx + u\eta dt, \quad dy = dt \quad (37)$$

this system (36) transforms into **ordinary differential equation** (see (29b) for comparison)

$$s_{yy} = \psi(e^s, s_y), \quad (38)$$

where $s = \ln \eta$. Integrability of this system (36) looks simpler than of the Calogero equation (1), but the ODE (38) is more complicated and reduction from (36) into (1) is not so obvious $\eta = \eta(u_x)$, where function η is solution of another ODE

$$\psi(\eta, g)d\eta = \eta g dg.$$

In next article [6] we show that one particular case of this two-component generalization (36), when $\psi = \frac{1}{2}(\eta^2 + u_x^2)$ (motion of dark matter in Universe, see [7]) is closely related with nonlinear Shrodinger – Maxwell-Bloch hierarchy. In this case, system (36) has infinite set of local Hamiltonian structures, commuting flows and conservation laws and can be written in Monge-Ampere form. Alternative interpretation of this special case was presented in [8] from nonlinear optics point of view.

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